

# Analysis of the SDFEM in a modified streamline diffusion norm for singularly perturbed convection diffusion problems<sup>☆</sup>

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## Abstract

In this paper we analyze a streamline diffusion finite element method (SDFEM) for a model singularly perturbed convection diffusion problem. To put insight into the influences of the stabilization parameter on SDFEM solutions, we discuss how to obtain the uniform estimates of errors in the streamline diffusion norm. By decreasing the standard stabilization parameters properly near the exponential layers, we obtain the uniform estimates in a norm, which is stronger than the  $\varepsilon$ -energy norm and weaker than the standard streamline diffusion norm.

*Keywords:* convection-diffusion problem, SDFEM, streamline diffusion norm

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## 1. Introduction

Consider the following singularly perturbed boundary value problem:

$$\begin{aligned} -\varepsilon \Delta u + \mathbf{b} \cdot \nabla u + cu &= f \quad \text{in } \Omega = (0, 1)^2, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where  $\varepsilon \ll |\mathbf{b}|$  is a positive parameter,  $\mathbf{b}(x, y) = (b_1(x, y), b_2(x, y))^T$ , and  $c(x, y)$  and  $f(x, y)$  are supposed sufficiently smooth. Also we assume that

$$b_1(x, y) \geq \beta_1 > 0, b_2(x, y) \geq \beta_2 > 0, c(x, y) \geq 0 \quad \text{on } \bar{\Omega},$$

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and

$$c(x, y) - \frac{1}{2} \nabla \cdot \mathbf{b}(x, y) \geq \mu_0 > 0 \quad \text{on } \bar{\Omega},$$

where  $\beta_1$ ,  $\beta_2$ , and  $\mu_0$  are some constants. These assumptions ensure that problem (1) has a unique solution in  $H_0^1(\Omega) \cap H^2(\Omega)$  for all  $f \in L^2(\Omega)$  (see, e.g., [1, 2]). In general there exist two exponential layers of width  $O(\varepsilon \ln(1/\varepsilon))$  at the sides  $x = 1$  and  $y = 1$  for the solution to problem (1).

For the convection-diffusion problem, we can obtain discrete solutions with satisfactory stability and accuracy by means of stabilized methods and a priori adapted meshes (see [3, 4]), for example, the streamline diffusion finite element method (SDFEM) [5] and a Shishkin mesh [6]. For the SDFEM on Shishkin rectangular meshes, convergence properties have been widely studied and the reader is referred to [7, 1, 8, 9, 10] and references therein.

It is easy to obtain uniform bounds of  $u - u^N$  in the  $\varepsilon$ -energy norm defined in (3) (see [7, 1]), where  $u$  is the solution to problem (1) and  $u^N$  is the SDFEM solution. Compared with the  $\varepsilon$ -energy norm, the streamline diffusion norm  $\|\cdot\|_{SD}$  defined in (4) is more proper to measure the energy properties of the SDFEM solution, which is derived from the bilinear form of the SDFEM. Nevertheless, it is impossible to obtain a uniform bound of  $\|u - u^I\|_{SD}$  where  $u^I$  is the interpolant of the solution  $u$  from the finite element space of piecewise bilinears, as can be seen by a simple one-dimensional example. The reason lies in the estimates of the term  $\sum \|\delta^{1/2} \nabla(u - u^I)\|$  in  $\|u - u^I\|_{SD}$ : there is always a factor  $\varepsilon^{-1}$ , which can not be balanced by the stabilization parameter  $\delta = O(N^{-1})$  ( $N$  is the mesh parameter).

In this paper, by decreasing the stabilization parameter near the exponential layers, we obtain uniform bounds of  $u - u^N$  in a modified streamline diffusion norm which is stronger than  $\varepsilon$ -energy norm but weaker than the standard streamline diffusion norm. With this modification of stabilization parameters, numerical stability of the SDFEM is preserved, as can be observed from numerical tests. Besides, we obtain the following uniform local estimates:

$$\|u - u^N\|_{SD; \Omega_s} \leq CN^{-3/2}, \quad \|u - u^N\|_{\varepsilon; \Omega_s} \leq C(\varepsilon^{1/2} N^{-1} + N^{-2} \ln^2 N),$$

where  $\Omega_s$  can be seen in Figure 1.

Here is the outline of this article. In §2 we give some a priori information for the solution of (1), then introduce the Shishkin meshes, a streamline diffusion finite element method on these meshes and our new stabilization

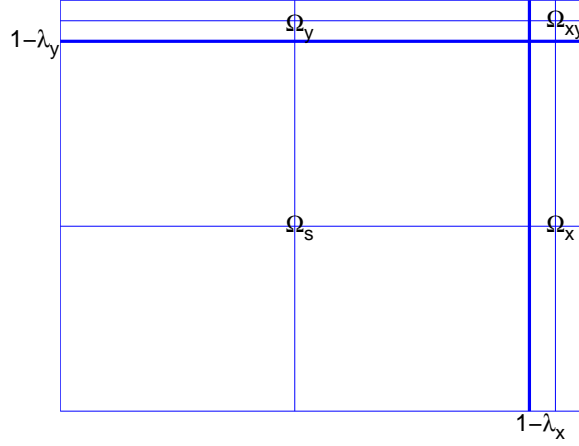


Figure 1: Shishkin meshes

parameters. In §3 we obtain the global and local estimates. Finally, some numerical results are presented in §4.

Throughout this paper,  $C$  will denote a generic positive constant, not necessarily the same at each occurrence, which is independent of  $\varepsilon$  and of the mesh parameter  $N$ .

## 2. The SDFEM on Shishkin meshes

For the convenience of reading we will present some basic knowledge in this section including the Shishkin meshes, the SDFEM and some assumptions.

### 2.1. Shishkin meshes

We use the *Shishkin* meshes to discretize (1), that is, there are both  $N$  (a positive even integer) mesh intervals in  $x$ - and  $y$ -direction which amass in the layer regions. For this purpose we assume that  $\varepsilon \leq N^{-1}$  and define the parameters

$$\lambda_x := \rho \frac{\varepsilon}{\beta_1} \ln N, \quad \lambda_y := \rho \frac{\varepsilon}{\beta_2} \ln N$$

where  $\rho = 2.5$ .

The domain  $\Omega$  is separated into four parts as  $\bar{\Omega} = \Omega_s \cup \Omega_x \cup \Omega_y \cup \Omega_{xy}$  (see Figure 1), where

$$\begin{aligned}\Omega_s &:= [0, 1 - \lambda_x] \times [0, 1 - \lambda_y], & \Omega_x &:= [1 - \lambda_x, 1] \times [0, 1 - \lambda_y], \\ \Omega_y &:= [0, 1 - \lambda_x] \times [1 - \lambda_y, 1], & \Omega_{xy} &:= [1 - \lambda_x, 1] \times [1 - \lambda_y, 1].\end{aligned}$$

Define

$$x_i = \begin{cases} 2i(1 - \lambda_x)/N, & \text{for } i = 0, \dots, N/2, \\ 1 - 2(N - i)\lambda_x/N, & \text{for } i = N/2 + 1, \dots, N, \end{cases}$$

and

$$y_j = \begin{cases} 2j(1 - \lambda_y)/N, & \text{for } j = 0, \dots, N/2, \\ 1 - 2(N - j)\lambda_y/N, & \text{for } j = N/2 + 1, \dots, N. \end{cases}$$

Draw lines through these mesh points parallel to the  $x$ -axis and  $y$ -axis, and the domain  $\Omega$  is dissected into rectangles. This yields a triangulation of  $\Omega$  denoted by  $\mathcal{T}_N$  (see Figure 1). If  $K = K_{ij} := [x_i, x_{i+1}] \times [y_j, y_{j+1}]$ , the mesh sizes  $h_{x,K} := x_{i+1} - x_i$  and  $h_{y,K} := y_{j+1} - y_j$  satisfy

$$h_{x,K} = \begin{cases} H_x := \frac{1 - \lambda_x}{N/2}, & \text{for } i = 0, \dots, N/2 - 1, \\ h_x := \frac{\lambda_x}{N/2}, & \text{for } i = N/2, \dots, N - 1, \end{cases}$$

and

$$h_{y,K} = \begin{cases} H_y := \frac{1 - \lambda_y}{N/2}, & \text{for } j = 0, \dots, N/2 - 1, \\ h_y := \frac{\lambda_y}{N/2}, & \text{for } j = N/2, \dots, N - 1. \end{cases}$$

## 2.2. The streamline diffusion finite element method

For any subdomain  $D$  of  $\Omega$ , denote the standard (semi-)norms in  $H^1(D)$  and  $L^2(D)$  by  $|\cdot|_{1,D}$  and  $\|\cdot\|_D$  respectively. If  $D = \Omega$  then we remove  $\Omega$  from the notation.

Let  $V := H_0^1(\Omega)$ . A variational formulation of problem (1) reads as

$$\begin{cases} \text{Find } u \in V \text{ such that for all } v \in V \\ \varepsilon(\nabla u, \nabla v) + (\mathbf{b} \cdot \nabla u + cu, v) = (f, v). \end{cases} \quad (2)$$

Obviously there is a unique solution of the weak formulation (2) by Lax-Milgram Lemma.

On the Shishkin meshes in the above subsection, we define a  $C^0$  bilinear finite element space as follows:

$$V^N := \{v^N \in C(\bar{\Omega}) : v^N|_{\partial\Omega} = 0 \text{ and } v^N|_K \in Q_1(K), \forall K \in \mathcal{T}_N\}.$$

The standard Galerkin discretisation of (2) reads:

$$\begin{cases} \text{Find } u^N \in V^N \text{ such that for all } v^N \in V^N \\ a_{Gal}(u^N, v^N) = (f, v^N), \end{cases}$$

where

$$a_{Gal}(u^N, v^N) = \varepsilon(\nabla u^N, \nabla v^N) + (\mathbf{b} \cdot \nabla u^N + cu^N, v^N).$$

An energy norm associated with  $a_{Gal}(\cdot, \cdot)$  reads:

$$\|v^N\|_\varepsilon^2 := \varepsilon|v^N|_1^2 + \mu_0\|v^N\|^2. \quad (3)$$

The SDFEM adds a stabilization term in a consistent way to the standard Galerkin discretisation and it reads:

$$\begin{cases} \text{Find } u^N \in V^N \text{ such that for all } v^N \in V^N, \\ a_{SD}(u^N, v^N) = (f, v^N) + \sum_{K \subset \Omega} (f, \delta \mathbf{b} \cdot \nabla v^N)_K, \end{cases}$$

where

$$a_{SD}(u^N, v^N) = a_{Gal}(u^N, v^N) + a_{stab}(u^N, v^N)$$

and

$$a_{stab}(u^N, v^N) = \sum_{K \subset \Omega} (-\varepsilon \Delta u^N + \mathbf{b} \cdot \nabla u^N + cu^N, \delta \mathbf{b} \cdot \nabla v^N)_K.$$

Note that  $\Delta u^N = 0$  in  $K$  for  $u^N|_K \in Q_1(K)$ . Here  $\delta = \delta(x, y)$  is a user-chosen parameter which will be defined later. We define the streamline diffusion norm (SD norm) associated with  $a_{SD}(\cdot, \cdot)$ :

$$\|v^N\|_{SD}^2 := \varepsilon|v^N|_1^2 + \mu_0\|v^N\|^2 + \sum_{K \subset \Omega} \|\delta^{1/2} \mathbf{b} \cdot \nabla v^N\|_K^2. \quad (4)$$

Set

$$x_t := 1 - \lambda_x, \quad x_s := x_t - H_x, \quad y_t := 1 - \lambda_y, \quad y_s := y_t - H_y$$

and

$$\Omega_{s,\varepsilon} = (0, x_s) \times (0, y_s), \quad \Omega_{s,\varepsilon}^c = \Omega_s \setminus \Omega_{s,\varepsilon}.$$

Note that  $\text{meas}(\Omega_{s,\varepsilon}^c) \leq CN^{-1}$ . For uniform estimates in the SD norm, we set

$$\xi(x) = \begin{cases} 1 & x \in [0, x_s] \\ \frac{x_t - x}{H_x} & x \in [x_s, x_t], \end{cases} \quad \eta(y) = \begin{cases} 1 & y \in [0, y_s] \\ \frac{y_t - y}{H_y} & y \in [y_s, y_t] \end{cases}$$

and

$$\delta(x, y) := \begin{cases} C^* N^{-1} \xi(x) \eta(y) & \text{if } (x, y) \in \Omega_s, \\ 0 & \text{if } (x, y) \in \Omega \setminus \Omega_s, \end{cases} \quad (5)$$

where  $C^*$  is a properly defined positive constant (see [4, Lemma 3.25]).

**Remark 1.** The definition (5) is different from the usual one (see [4])

$$\delta(x, y) := \begin{cases} C^* N^{-1}, & \text{if } (x, y) \in \Omega_s, \\ 0, & \text{otherwise.} \end{cases} \quad (6)$$

In fact they are different from each other only in  $\Omega_{s,\varepsilon}^c$ . The uniform interpolation estimates in Lemma 2 depend on the definition in  $\Omega_{s,\varepsilon}^c$  of (5) (see (17) and (14)–(16)).

Note the SD norm (4) is stronger than the  $\varepsilon$ -energy norm (3). Clearly, the SD norm with (5) is a little weaker than one with (6).

For any subdomain  $D$  of  $\Omega$ , notations  $\|\cdot\|_{\varepsilon; D}$  and  $\|\cdot\|_{SD; D}$  mean that the integrations in (3) and (4) are restricted in  $D$ .

### 2.3. The regularity results and interpolation bounds

In this paper we always assume that the solution of (1) consists of a regular solution component and various layer parts as follows.

**Assumption 1.** For our analysis we shall assume that the solution of (1) can be decomposed as

$$u = S + E_1 + E_2 + E_{12}. \quad (7)$$

For all  $(x, y) \in \bar{\Omega}$ , the regular part  $S$  and the layer terms  $E_1$ ,  $E_2$  and  $E_{12}$  satisfy

$$\begin{aligned} |\partial_{x,y}^{i+j} S| &\leq C \quad \text{for } 0 \leq i + j \leq 3, \\ |\partial_{x,y}^{i+j} E_1| &\leq C \varepsilon^{-i} e^{-\beta_1(1-x)/\varepsilon} \quad \text{for } 0 \leq i, j \leq 2, \\ |\partial_{x,y}^{i+j} E_2| &\leq C \varepsilon^{-j} e^{-\beta_2(1-y)/\varepsilon} \quad \text{for } 0 \leq i, j \leq 2, \\ |\partial_{x,y}^{i+j} E_{12}| &\leq C \varepsilon^{-(i+j)} e^{-(\beta_1(1-x) + \beta_2(1-y))/\varepsilon} \quad \text{for } 0 \leq i, j \leq 2, \end{aligned} \quad (8)$$

where  $\partial_{x,y}^{i+j}v := \frac{\partial^{i+j}v}{\partial x^i \partial y^j}$ . Furthermore, assume that  $S \in H^3(\Omega)$  with

$$\|S\|_{H^3(\Omega)} \leq C.$$

**Remark 2.** The conditions that ensure the above assumption valid can be found in [11, Theorem 5.1] and [12, 13].

The following bounds will be frequently used.

**Lemma 1.** Let  $u^I$  and  $E^I$  denote the piecewise bilinear interpolation of  $u$  and  $E$  on the Shishkin mesh  $\mathcal{T}_N$  respectively, where  $E = E_1 + E_2 + E_{12}$ . Suppose that  $u$  satisfies Assumption 1. Then we have

$$\begin{aligned} \|u - u^I\|_{L^\infty(K)} &\leq \begin{cases} C \max\{N^{-2}, N^{-\rho}\} & \text{if } K \subset \Omega_s, \\ CN^{-2} \ln^2 N & \text{otherwise,} \end{cases} \\ \|\nabla(u - u^I)\|_{L^1(\Omega_s)} &\leq CN^{-1}, \quad \|\nabla E^I\|_{L^1(\Omega_s)} \leq CN^{-\rho}. \end{aligned}$$

*Proof.* See the details in [7, Theorem 4.2] for the first estimate. The reader is referred to [1, Lemma 3.2] for the remained bounds.  $\square$

### 3. Interpolation and error estimates in the SD norm

**Lemma 2.** Let Assumption 1 hold true and  $\delta$  be defined in (5), we have

$$\|u - u^I\|_{SD} \leq CN^{-1} \ln N.$$

*Proof.* From (3) and (4), we obtain

$$\begin{aligned} \|u - u^I\|_{SD}^2 &= \|u - u^I\|_\varepsilon^2 + \sum_{K \subset \Omega} (\mathbf{b} \cdot \nabla(u - u^I), \delta \mathbf{b} \cdot \nabla(u - u^I))_K \\ &=: \text{I} + \text{II}. \end{aligned}$$

The bound of I can be found in [7, Theorem 4.3], that is,

$$|\text{I}| \leq CN^{-2} \ln^2 N. \tag{9}$$

Using the decomposition (7), we have

$$u - u^I = S - S^I + E - E^I,$$

where  $E = E_1 + E_2 + E_{12}$ . Note that  $\delta = 0$  for  $(x, y) \in \Omega \setminus \Omega_s$ . Then, we rewrite II as follows:

$$\begin{aligned} \text{II} &= \sum_{K \subset \Omega_s} (\mathbf{b} \cdot \nabla(S - S^I), \delta \mathbf{b} \cdot \nabla(u - u^I))_K + \sum_{K \subset \Omega_s} (-\mathbf{b} \cdot \nabla E^I, \delta \mathbf{b} \cdot \nabla(u - u^I))_K \\ &\quad + \sum_{K \subset \Omega_s} (\mathbf{b} \cdot \nabla E, \delta \mathbf{b} \cdot \nabla(S - S^I - E^I))_K + \sum_{K \subset \Omega_s} (\mathbf{b} \cdot \nabla E, \delta \mathbf{b} \cdot \nabla E)_K \\ &=: \text{II}_1 + \text{II}_2 + \text{II}_3 + \text{II}_4. \end{aligned}$$

We will estimate II term by term.

Note that  $\delta \leq CN^{-1}$  for  $(x, y) \in \Omega_s$ . By standard interpolation theories, the inequalities (8) and Lemma 1 we have

$$|\text{II}_1| \leq CN^{-1} \|\nabla(S - S^I)\|_{L^\infty(\Omega_s)} \|\nabla(u - u^I)\|_{L^1(\Omega_s)} \leq CN^{-3}. \quad (10)$$

Inverse estimates [14, Theorem 3.2.6] and Assumption 1 yield

$$\|\nabla E^I\|_{L^\infty(\Omega_s)} \leq CN \|E^I\|_{L^\infty(\Omega_s)} \leq CN^{1-\rho}. \quad (11)$$

Lemma 1 and the bound (11) yield

$$|\text{II}_2| \leq CN^{-1} \|\nabla E^I\|_{L^\infty(\Omega_s)} \|\nabla(u - u^I)\|_{L^1(\Omega_s)} \leq CN^{-(1+\rho)}, \quad (12)$$

and

$$\begin{aligned} |\text{II}_3| &\leq CN^{-1} \|\nabla E\|_{L^1(\Omega_s)} (\|\nabla(S - S^I)\|_{L^\infty(\Omega_s)} + \|\nabla E^I\|_{L^\infty(\Omega_s)}) \\ &\leq C(N^{-2} + N^{-\rho}) \|\nabla E\|_{L^1(\Omega_s)} \leq CN^{-(2+\rho)}. \end{aligned} \quad (13)$$

To bound the term  $\text{II}_4$ , we present the following estimates first. Note that  $0 \leq xe^{-x} \leq e^{-1}$  for  $x \geq 0$ . Then we have

$$\begin{aligned} \int_0^{x_s} e^{-\frac{2\beta_1(1-x)}{\varepsilon}} dx &= \int_0^{x_s} e^{-\frac{2\beta_1(1-x_t)}{\varepsilon}} e^{-\frac{2\beta_1(x_t-x)}{\varepsilon}} dx \\ &= N^{-2\rho} \frac{\varepsilon}{2\beta_1} \left( e^{-\frac{2\beta_1 H_x}{\varepsilon}} - e^{-\frac{2\beta_1 x_t}{\varepsilon}} \right) \\ &\leq N^{-2\rho} \left( \frac{\varepsilon}{2\beta_1} \right)^2 H_x^{-1} \cdot \frac{2\beta_1 H_x}{\varepsilon} e^{-\frac{2\beta_1 H_x}{\varepsilon}} \\ &\leq C\varepsilon^2 N^{1-2\rho}, \end{aligned} \quad (14)$$



and

$$\begin{aligned}
\int_{x_s}^{x_t} e^{-\frac{2\beta_1(1-x)}{\varepsilon}} \frac{x_t - x}{H_x} dx &= \int_{x_s}^{x_t} e^{-\frac{2\beta_1(1-x_t)}{\varepsilon}} e^{-\frac{2\beta_1(x_t-x)}{\varepsilon}} \frac{x_t - x}{H_x} dx \\
&\stackrel{\xi=x_t-x}{=} N^{-2\rho} H_x^{-1} \int_0^{H_x} e^{-\frac{2\beta_1\xi}{\varepsilon}} \xi d\xi \\
&= N^{-2\rho} H_x^{-1} \left( \frac{\varepsilon}{2\beta_1} \right)^2 \left( -\frac{2\beta_1 H_x}{\varepsilon} e^{-\frac{2\beta_1 H_x}{\varepsilon}} + 1 - e^{-\frac{2\beta_1 H_x}{\varepsilon}} \right) \\
&\leq C\varepsilon^2 N^{1-2\rho}.
\end{aligned} \tag{15}$$

Similarly, we have

$$\int_0^{y_s} e^{-\frac{2\beta_2(1-y)}{\varepsilon}} dy \leq C\varepsilon^2 N^{1-2\rho}, \quad \int_{y_s}^{y_t} e^{-\frac{2\beta_2(1-y)}{\varepsilon}} \frac{y_t - y}{H_y} dy \leq C\varepsilon^2 N^{1-2\rho}. \tag{16}$$

According to the definition of  $\delta$  in (5), we split  $\Pi_4$  into two parts. Then

$$\begin{aligned}
|\Pi_4| &= \left( \sum_{K \subset \Omega_{s,\varepsilon}} + \sum_{K \subset \Omega_{s,\varepsilon}^c} \right) |(\mathbf{b} \cdot \nabla E, \delta \mathbf{b} \cdot \nabla E)_K| \\
&\leq CN^{-1} \|\nabla E\|_{\Omega_{s,\varepsilon}}^2 + C \iint_{\Omega_{s,\varepsilon}^c} \varepsilon^{-2} \left( e^{-\frac{2\beta_1(1-x)}{\varepsilon}} + e^{-\frac{2\beta_2(1-y)}{\varepsilon}} \right) \delta(x, y) dx dy \\
&\leq CN^{-2\rho},
\end{aligned} \tag{17}$$

where we have used (14)–(16) and direct calculations.

Collecting (10), (12), (13) and (17), we obtain

$$|\Pi| \leq CN^{-3}. \tag{18}$$

Combine (9) and (18), then we are done.  $\square$

**Remark 3.** We can obtain local estimates in  $\Omega_s$  in the same way as in Lemma 2. If Assumption 1 holds true, we have

$$\|u - u^I\|_{\varepsilon; \Omega_s}^2 = \varepsilon \|\nabla(u - u^I)\|_{\Omega_s}^2 + \mu_0 \|u - u^I\|_{\Omega_s}^2 \leq C(\varepsilon N^{-2} + N^{-4}). \tag{19}$$

Note that analysis of  $\varepsilon \|\nabla(u - u^I)\|_{\Omega_s}^2$  is similar to one of  $\Pi$  in Lemma 2. Moreover, if  $\delta$  is defined in (5), the bounds (19) and (18) yield

$$\begin{aligned}
\|u - u^I\|_{SD; \Omega_s}^2 &= \|u - u^I\|_{\varepsilon; \Omega_s}^2 + \sum_{K \subset \Omega} (\mathbf{b} \cdot \nabla(u - u^I), \delta \mathbf{b} \cdot \nabla(u - u^I))_K \\
&\leq CN^{-3}.
\end{aligned} \tag{20}$$

**Lemma 3.** *Let Assumption 1 hold true and  $\delta$  be defined in (5) or in (6), then we have*

$$\|u^I - u^N\|_{SD} \leq C(\varepsilon N^{-3/2} + N^{-2} \ln^2 N).$$

*Proof.* See the details in [1, Theorem 4.5]. □

By the above lemmas we have the following theorem.

**Theorem 1.** *Let Assumption 1 hold true. If  $\delta$  is defined in (5), we have*

$$\|u - u^N\|_{SD} \leq CN^{-1} \ln N, \quad (21)$$

$$\|u - u^N\|_{SD; \Omega_s} \leq CN^{-3/2}. \quad (22)$$

*If  $\delta$  is defined in (5) or in (6), we have*

$$\|u - u^N\|_{\varepsilon; \Omega_s} \leq C(\varepsilon^{1/2} N^{-1} + N^{-2} \ln^2 N). \quad (23)$$

*Proof.* Combining Lemmas 2 and 3, we obtain (21). Lemma 3 and (20) yield (22). Note that  $\|u^I - u^N\|_{\varepsilon} \leq \|u^I - u^N\|_{SD}$ . Thus from (19) and Lemma 3, we have (23). □

#### 4. Numerical results

In this section we give the numerical results that appear to support our theoretical results. Errors and convergence rates in different norms are presented. Numerical experiments show that our new stabilization parameter preserves high accuracy and numerical stability as the standard one.

All calculations were carried out by using Intel visual Fortran 11. The discrete problems were solved by the nonsymmetric iterative solver GMRES(c.f. e.g., [15, 16]).

**Problem.**

$$\begin{aligned} -\varepsilon \Delta u + 2u_x + u_y + u &= f(x, y) \quad \text{in } \Omega = (0, 1)^2, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \quad (24)$$

where the right-hand side  $f$  is chosen such that

$$u(x, y) = 2 \sin x \left(1 - e^{-\frac{2(1-x)}{\varepsilon}}\right) y^2 \left(1 - e^{-\frac{(1-y)}{\varepsilon}}\right)$$

is the exact solution.

Table 1:  $\delta$  in (6)

$N$	$\varepsilon = 10^{-4}$		$\varepsilon = 10^{-6}$		$\varepsilon = 10^{-8}, 10^{-10}, \dots, 10^{-16}$	
	$\ u - u^N\ _{\varepsilon; \Omega_s}$	Rate	$\ u - u^N\ _{\varepsilon; \Omega_s}$	Rate	$\ u - u^N\ _{\varepsilon; \Omega_s}$	Rate
8	$7.79 \times 10^{-3}$	1.78	$7.65 \times 10^{-3}$	1.84	$7.64 \times 10^{-3}$	1.84
16	$2.26 \times 10^{-3}$	1.87	$2.13 \times 10^{-3}$	2.14	$2.13 \times 10^{-3}$	2.14
32	$6.19 \times 10^{-4}$	1.46	$4.85 \times 10^{-4}$	2.05	$4.83 \times 10^{-4}$	2.06
64	$2.25 \times 10^{-4}$	1.16	$1.17 \times 10^{-4}$	1.97	$1.16 \times 10^{-4}$	2.03
128	$1.01 \times 10^{-4}$	1.04	$3.00 \times 10^{-5}$	1.81	$2.85 \times 10^{-5}$	2.01
256	$4.87 \times 10^{-5}$	1.01	$8.55 \times 10^{-6}$	1.52	$7.07 \times 10^{-6}$	2.00
512	$2.41 \times 10^{-5}$	—	$2.99 \times 10^{-6}$	—	$1.77 \times 10^{-6}$	—

Table 2:  $\delta$  in (5)

$N$	$\varepsilon = 10^{-4}$		$\varepsilon = 10^{-6}$		$\varepsilon = 10^{-8}, 10^{-10}, \dots, 10^{-16}$	
	$\ u - u^N\ _{\varepsilon; \Omega_s}$	Rate	$\ u - u^N\ _{\varepsilon; \Omega_s}$	Rate	$\ u - u^N\ _{\varepsilon; \Omega_s}$	Rate
8	$1.08 \times 10^{-2}$	1.82	$1.07 \times 10^{-2}$	1.85	$1.07 \times 10^{-2}$	1.85
16	$3.05 \times 10^{-3}$	2.08	$2.96 \times 10^{-3}$	2.27	$2.96 \times 10^{-3}$	2.27
32	$7.23 \times 10^{-4}$	1.63	$6.12 \times 10^{-4}$	2.19	$6.11 \times 10^{-4}$	2.20
64	$2.34 \times 10^{-4}$	1.21	$1.34 \times 10^{-4}$	2.06	$1.33 \times 10^{-4}$	2.12
128	$1.01 \times 10^{-4}$	1.05	$3.21 \times 10^{-5}$	1.87	$3.06 \times 10^{-5}$	2.06
256	$4.87 \times 10^{-5}$	1.01	$8.77 \times 10^{-6}$	1.55	$7.07 \times 10^{-6}$	2.00
512	$2.41 \times 10^{-5}$	—	$3.01 \times 10^{-6}$	—	$1.81 \times 10^{-6}$	—

The errors in Tables 1–4 are measured as follows

$$e_{SD}^N := \left( \sum_{K \subset \Omega} \|u - u^N\|_{SD,K}^2 \right)^{1/2}, \quad e_{\varepsilon}^N := \left( \sum_{K \subset \Omega} \|u - u^N\|_{\varepsilon,K}^2 \right)^{1/2},$$

where  $u^N$  is the SDFEM solution.

The corresponding rates of convergence  $p^N$  are computed from the formula

$$p^N = \frac{\ln e^N - \ln e^{2N}}{\ln 2}, \quad (25)$$

where  $e^N$  can be  $e_{SD}^N$ ,  $e_{\varepsilon; \Omega_s}^N$  or  $e_{SD; \Omega_s}^N$ .

Tables 1 and 2 present the errors and convergence rates of  $\|u - u^N\|_{\varepsilon; \Omega_s}$ , which support the theoretical bound (23). Moreover, we observe that if  $\varepsilon \leq N^{-2}$  the convergence order of  $\|u - u^N\|_{\varepsilon; \Omega_s}$  is almost 2. If  $\varepsilon \geq N^{-2}$ , maybe  $\varepsilon^{1/2} N^{-1}$  dominates the bound of  $\|u - u^N\|_{\varepsilon; \Omega_s}$ .

Table 3 gives the errors and convergence rates of  $\|u - u^N\|_{SD; \Omega_s}$ , which show that the convergence order of  $\|u - u^N\|_{SD; \Omega_s}$  is  $3/2$ .

Table 3:  $\varepsilon = 10^{-4}, 10^{-6}, \dots, 10^{-16}$ 

$N$	$\delta$ in (6)		$\delta$ in (5)	
	$\ u - u^N\ _{SD; \Omega_s}$	Rate	$\ u - u^N\ _{SD; \Omega_s}$	Rate
8	$1.80 \times 10^{-1}$	1.50	$1.37 \times 10^{-1}$	1.29
16	$6.38 \times 10^{-2}$	1.50	$5.59 \times 10^{-2}$	1.41
32	$2.25 \times 10^{-2}$	1.50	$2.11 \times 10^{-2}$	1.45
64	$7.96 \times 10^{-3}$	1.50	$7.70 \times 10^{-3}$	1.48
128	$2.81 \times 10^{-3}$	1.50	$2.77 \times 10^{-3}$	1.49
256	$9.94 \times 10^{-4}$	1.50	$9.86 \times 10^{-4}$	1.49
512	$3.52 \times 10^{-4}$	—	$3.50 \times 10^{-4}$	—

Table 4:  $\varepsilon = 10^{-4}, 10^{-6}, \dots, 10^{-16}$ 

$N$	$\delta$ in (6)				$\delta$ in (5)			
	$\ u - u^N\ _\varepsilon$	Rate	$\ u - u^N\ _{SD}$	Rate	$\ u - u^N\ _\varepsilon$	Rate	$\ u - u^N\ _{SD}$	Rate
8	$4.19 \times 10^{-1}$	0.63	$4.56 \times 10^{-1}$	0.71	$4.19 \times 10^{-1}$	0.63	$4.41 \times 10^{-1}$	0.67
16	$2.71 \times 10^{-1}$	0.70	$2.79 \times 10^{-1}$	0.73	$2.77 \times 10^{-1}$	0.70	$2.79 \times 10^{-1}$	0.72
32	$1.67 \times 10^{-1}$	0.75	$1.68 \times 10^{-1}$	0.76	$1.67 \times 10^{-1}$	0.75	$1.68 \times 10^{-1}$	0.75
64	$9.93 \times 10^{-2}$	0.78	$9.96 \times 10^{-2}$	0.78	$9.93 \times 10^{-2}$	0.78	$9.96 \times 10^{-2}$	0.78
128	$5.78 \times 10^{-2}$	0.81	$5.78 \times 10^{-2}$	0.81	$5.78 \times 10^{-2}$	0.81	$5.78 \times 10^{-2}$	0.81
256	$3.30 \times 10^{-2}$	0.83	$3.30 \times 10^{-2}$	0.83	$3.30 \times 10^{-2}$	0.83	$3.30 \times 10^{-2}$	0.83
512	$1.85 \times 10^{-2}$	—	$1.85 \times 10^{-2}$	—	$1.85 \times 10^{-2}$	—	$1.85 \times 10^{-2}$	—

In Table 4, the errors and convergence rates for  $\|u - u^N\|_{SD}$  and  $\|u - u^N\|_\varepsilon$  are displayed, which support (21). We observe similar bounds and convergence orders of  $u - u^N$  with  $\delta$  defined in (5) or (6).

Plots 2—5 show that with the new stabilization parameter  $\delta$ , the SDFEM solutions still preserve high accuracy and numerical stability. Figures 2 and 3 present pointwise errors in the computational domain  $\Omega$ , Figures 4 and 5 in one exponential layer. These plots show that there are no visible oscillations in the SDFEM solutions with  $\delta$  in (5), as the SDFEM solutions with  $\delta$  in (6).

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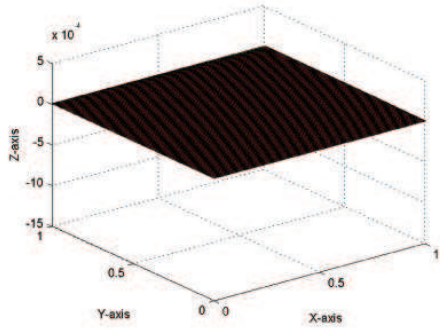


Figure 2: Pointwise errors with  $\delta$  in (6)

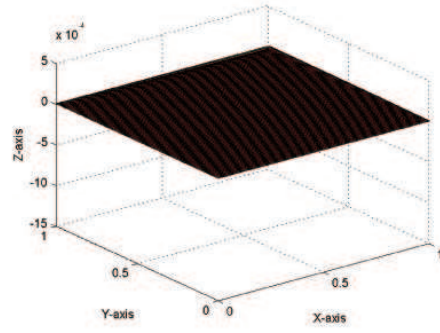


Figure 3: Pointwise errors with  $\delta$  in (5)

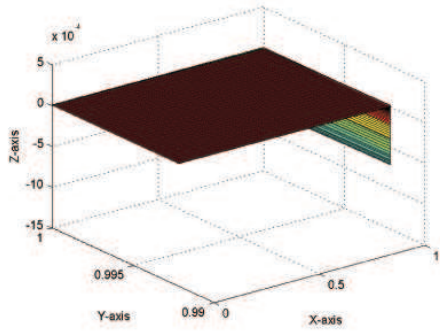


Figure 4: Pointwise errors in exponential layers with  $\delta$  in (6)

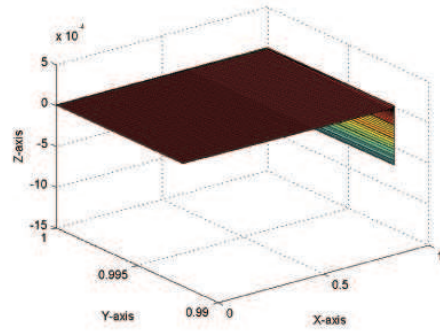


Figure 5: Pointwise errors in exponential layers with  $\delta$  in (5)

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